# Pools of ions trapped below the surface of superfluid helium: modes of response in a steady vertical magnetic field



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# Pools of ions trapped below the surface of superfluid helium: modes of response in a steady vertical magnetic field

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**Abstract.** The paper is concerned with two-dimensional classical fluids and solids composed of charged particles that interact only through Coulomb forces. An example of this type of system, and one with which the paper is particularly concerned, is a circular pool of either positive or negative ions trapped just below the free surface of superfluid helium. A calculation is presented of the modes of response of such a system in its own plane when exposed to an oscillating electric potential, especially in the long-wavelength limit and in the presence of a steady magnetic field applied in a direction normal to the plane of the system. The calculations are relevant to the experimental detection and study of these modes, which include shear modes in the crystal phase and viscous modes in the fluid phase.

## 1. Introduction

Pools of either positive or negative ions can be trapped just below the free surface of superfluid $^4$He (see, for example, Barenghi *et al* 1991). The pools form two-dimensional arrays of particles interacting through Coulomb forces; since the effective masses $M$ of the ions are high (about 30 $m_4$ for the positive ion and about 237 $m_4$ for the negative ion), the pools behave classically. The ions have in practice some thermally excited vertical motion, so that pools are not strictly two-dimensional. However, the vertical motion has an amplitude that is much smaller than the ionic separation and a frequency that is much higher than any with which we are concerned in this paper. For the purpose of the present paper therefore the pools can be treated as two-dimensional. The details of the trapping mechanism and the nature of the ions are described Barenghi *et al*, but they need not concern us here.

The pools can exist as two-dimensional Coulomb crystals or as two-dimensional fluids, depending on the temperature (Vinen *et al* 1994). They may also exist in a hexatic phase in a range of temperature between the crystal and fluid phases (Nelson and Halperin 1979). This paper is concerned with calculations of the expected response of these pools in the horizontal plane when they are exposed to an oscillating electric potential. We consider the response of both the crystal and fluid phases, especially in the presence of a steady vertical magnetic field. The calculations underlie various experimental investigations, particularly of shear-mode propagation in the pools, which are being published separately (Elliott *et al* 1995a). Although our calculations relate specifically to ion pools, they are applicable to other two-dimensional systems of this type: for example to *electron* pools trapped *above* the helium surface (see, for example, Grimes and Adams 1976).

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In the absence of the magnetic field there are two separate and independent modes of response: longitudinal plasma modes and transverse shear modes. The plasma modes, which were first observed by Ott-Rowland *et al* (1982) and studied in detail by Barenghi *et al* (1991), are essentially similar in all phases of the pool. If we neglect a small damping the transverse modes in the crystal phase have the dispersion relation

$$\omega_t^2(k) = \frac{\mu k^2}{n_0 M} \tag{1.1}$$

where $n_0$ is the areal number density in the pool and $\mu$ is the shear modulus of the crystal, given at a low temperature by (Bonsall and Maradudin 1977)

$$\mu = \frac{0.245 n_0^{3/2} e^2}{4\pi\epsilon_0}. \tag{1.2}$$

Transverse modes in the fluid phase are viscous waves with the dispersion relation

$$\omega_v = -\frac{i\eta k^2}{n_0 M} \tag{1.3}$$

where $\eta$ is the shear viscosity of the plasma. We can usefully think of the fluid as having an effective shear modulus equal to $-i\omega\eta$. In a hexatic phase the transverse modes are expected to be more complicated (Zippelius *et al* 1980), and we shall not consider them in this paper.

In the presence of a steady vertical magnetic field $B$ these modes are modified and the longitudinal and transverse modes become coupled through the action of the Lorentz force. Several studies of the effect on the longitudinal modes have been published (Mellor *et al* (1988) and Appleyard *et al* (1995) for the ion pools, and Glattli *et al* (1985) and Mast *et al* (1985) for the electron pools), but they have not taken account of any shear modulus. A brief discussion of the effect of a magnetic field on the transverse modes was included in a report of the observation of transverse modes in the crystal phase of the electron pools (Deville *et al* 1984), but not all aspects of the theoretical problem were explored. In this paper we shall present detailed calculations that take account of the shear moduli, for both the crystal and fluid phases. The effect on the longitudinal modes turns out to be small, and we shall concentrate our attention on the shear modes. The results of our calculations are important because they lead to experimental methods for the study of the shear modulus in the crystal phase (Elliott *et al* 1995a) and the shear viscosity in the fluid phase. These experimental methods may also provide a tool for a study of the anomalous dynamic properties that are expected to be associated with a hexatic phase. In practice the modes of the pools are excited by applying an oscillating potential to a set of electrodes surrounding the pool and detected by the potentials induced in a second set of electrodes. The fact that a magnetic field couples the transverse and longitudinal modes is important experimentally because otherwise these techniques could not be used in the study of the transverse modes. In practice a magnetic field adds only relatively small longitudinal components to the low-frequency shear modes with which we shall be particularly concerned in this paper, and we shall therefore refer to them simply as shear modes, even in the presence of the magnetic field.

We shall restrict our calculations to modes of response with wavenumbers that are small compared with the inverse interparticle spacing, so that, for example, we are not concerned with dispersion near the Brillouin zone boundary of the crystal. When we discuss the modes of response of a *bounded* pool (section 3), we shall also restrict our calculations to wavenumbers that are small compared with a length $h$, introduced later, which determines *inter alia* whether the restoring force on the ions resulting from a perturbation to the ion density is local or non-local. We shall neglect any effects due to ripplon coupling, except those associated with a ripplon-limited mobility; other effects due to ripplon coupling are in any case small in the case of the ion pools. Electron pools above the helium surface are more strongly affected by ripplon interactions, especially in the crystal phase, but we shall not take account of these effects here.

We shall first consider the modes of response of an *unbounded* ion pool, which are relatively straightforward (section 2). In section 3 we consider a circular *bounded* pool, such as is used in experiments. An accurate treatment of the bounded pool is possible only by using computational techniques, but we argue that modes of response with small wavenumbers can be treated analytically if we introduce effective boundary conditions at the edge of the pool. We summarise in section 4.

## 2. The modes of response of an unbounded pool

In practice the ion pools are situated between two horizontal metallic electrodes, and in this section we shall assume that such electrodes (of infinite extent) surround an unbounded pool. Let $2h$ be the separation between the electrodes and $d$ the separation between the plane of the pool and the lower electrode. The electrodes are assumed to have infinite electrical conductivity and to be effectively grounded, so that perturbations in the charge density in the pool do not lead to any perturbation in the electrode potentials. In this section we discuss explicitly only the response of a crystalline pool; as we have seen the response of a fluid pool can be obtained by the substitution $\mu \to -i\omega\eta$. We shall use cylindrical polar coordinates $(r, \theta, z)$, the ions being in the plane $z = 0$.

Let $\boldsymbol{u}$ be the vector field describing the ionic displacement associated with the mode. The linearized equation of motion of the ions in the crystal phase in the presence of a steady vertical uniform magnetic field $\boldsymbol{B}$ (pointing normal to the plane of the ions) is then

$$\ddot{\boldsymbol{u}} + \frac{1}{\tau}\dot{\boldsymbol{u}} = -\frac{e}{M}\nabla_{\perp}\phi + \frac{e}{M}\dot{\boldsymbol{u}} \times \boldsymbol{B} + \frac{\mu}{Mn_0}\nabla^2\boldsymbol{u} \tag{2.1}$$

where $\tau$ is the relaxation time associated with the finite ionic mobility, $\phi$ is the electrostatic potential in the plane of the ions, and $n_0$ is the equilibrium areal number density of the ions (assumed to be uniform). It is convenient to introduce two scalar fields $L$ and $T$ defined by

$$L = \nabla \cdot \boldsymbol{u} \qquad \text{and} \qquad T = \hat{z} \cdot (\nabla \times \boldsymbol{u}) \tag{2.2}$$

where $\hat{z}$ is the unit vector normal to the plane of the ions. Taking first the divergence and then the curl of equation (2.1) we obtain

$$\ddot{L} + \frac{1}{\tau}\dot{L} - \frac{\mu}{Mn_0}\nabla^2 L + \frac{e}{M}\nabla_{\perp}^2\phi = \frac{eB}{M}\dot{T} \tag{2.3}$$

where $\nabla_{\perp}^2 = \nabla^2 - \partial^2/\partial z^2$ and

$$\ddot{T} + \frac{1}{\tau}\dot{T} - \frac{\mu}{Mn_0}\nabla^2 T = -\frac{eB}{M}\dot{L}. \tag{2.4}$$

The linearized equation of continuity for fluctuations $\sigma$ in the ionic charge density is

$$\dot{\sigma} = -n_0 e \nabla \cdot \dot{u} = -n_0 e \dot{L}. \tag{2.5}$$

A solution of Laplace's equation appropriate to the symmetry of a circular ion pool is

$$\phi = \phi_{k,m} J_m(kr) \exp(\mathrm{i}m\theta) \exp(\pm kz). \tag{2.6}$$

As shown in, for example, Barenghi *et al* (1991), if the boundary conditions at the plates are satisfied, then the corresponding perturbation in the charge density is given by

$$\sigma = \sigma_{k,m} J_m(kr) \exp(\mathrm{i}m\theta) \tag{2.7}$$

where

$$\phi_{k,m} = \frac{F(k)}{2\epsilon_0 k}\sigma_{k,m} \qquad \text{and } F(k) = \frac{2\sinh(kd)\sinh\{k(2h-d)\}}{\sinh(2kh) + (\epsilon - 1)\sinh\{k(2h-d)\}\cosh(kd)}. \tag{2.8}$$

For simplicity, but without loss of any essential physics, we shall take $d = h$ and neglect the term in $(\epsilon - 1)$. Then

$$F(k) = \tanh kh. \tag{2.9}$$

Note also, for future reference, that if $k = \mathrm{i}k''$ is purely imaginary

$$F(k) = \mathrm{i} \tan k'' h. \tag{2.10}$$

Let us calculate the response of the pool to the externally applied potential

$$\Phi = \Phi_{k,m} J_m(kr) \exp(\mathrm{i}m\theta) \exp(-\mathrm{i}\omega t) \tag{2.11}$$

in the plane of the pool, so that the potential appearing in equations (2.1) and (2.3) is the sum of (2.6) and (2.11).

Tentatively, we assume that the resulting $L$ and $T$ take the forms

$$\begin{aligned} L &= L_{k,m} J_m(kr) \exp(\mathrm{i}m\theta) \exp(-\mathrm{i}\omega t) \\ T &= T_{k,m} J_m(kr) \exp(\mathrm{i}m\theta) \exp(-\mathrm{i}\omega t). \end{aligned} \tag{2.12}$$

Substituting into (2.3) and (2.4), we find that

$$\left[\omega^2 + \frac{\mathrm{i}\omega}{\tau} - \omega_l^2(k)\right] L_{k,m} - \mathrm{i}\omega\omega_c T_{k,m} = -\frac{ek^2}{M}\Phi_{k,m} \tag{2.13}$$

and

$$\left[\omega^2 + \frac{\mathrm{i}\omega}{\tau} - \omega_l^2(k)\right] T_{k,m} + \mathrm{i}\omega\omega_c L_{k,m} = 0 \tag{2.14}$$

where

$$\omega_l^2(k) = \frac{n_0 e^2 k}{2\epsilon_0 M}\tanh kh + \frac{\mu k^2}{n_0 M} \tag{2.15}$$

and, as before,

$$\omega_t^2(k) = \frac{\mu k^2}{n_0 M} \tag{2.16}$$

if $k$ is real, or

$$\omega_l^2(k) = -\frac{n_0 e^2 k''}{2\epsilon_0 M} \tan k''h - \frac{\mu k''^2}{n_0 M} \tag{2.15a}$$

and

$$\omega_t^2 = -\frac{\mu k''^2}{n_0 M} \tag{2.16a}$$

if $k$ is imaginary; and where

$$\omega_c = \frac{eB}{M}. \tag{2.17}$$

We confirm that solutions of the form (2.12) do exist. From equations (2.13) and (2.14) we obtain the responses $L_{k,m}$ and $T_{k,m}$ to drive the $\Phi_{k,m}$:

$$L_{k,m} = -\frac{ek^2}{M}\,\Phi_{k,m}\,\frac{\omega^2 - \omega_t^2(k) + \mathrm{i}\omega/\tau}{[\omega^2 - \omega_t^2(k) + \mathrm{i}\omega/\tau][\omega^2 - \omega_l^2(k) + \mathrm{i}\omega/\tau] - \omega^2\omega_c^2} \tag{2.18}$$

and

$$T_{k,m} = \frac{-\mathrm{i}\omega\omega_c}{[\omega^2 - \omega_t^2(k) + \mathrm{i}\omega/\tau]} L_{k,m}. \tag{2.19}$$

The dispersion relation for the modes of oscillation is given by the vanishing of the denominator of (2.18). In the limit of zero damping ($\tau \to \infty$) we find two branches, given by

$$2\omega_\pm^2(k) = \omega_l^2 + \omega_t^2 + \omega_c^2 \pm \left[\left(\omega_l^2 + \omega_t^2 + \omega_c^2\right)^2 - 4\omega_l^2\omega_t^2\right]^{1/2}. \tag{2.20}$$

In the absence of a magnetic field ($\omega_c = 0$) the two frequencies are real only if $k$ is real, and they reduce simply to

$$\omega_+ = \omega_l \qquad \omega_- = \omega_t. \tag{2.21}$$

These two dispersion relations are plotted in figure 1, for a typical value of the density $n_0$. In the presence of a magnetic field the two frequencies can be real not only for real values of $k$ but also, at least formally, over certain ranges of imaginary $k$. For the case $B = 1.2$ T, and for the same density as in figure 1, the exact forms of the dispersion relations (2.20) are plotted in figure 2. In practice, for the case of real $k$, $\omega_t \ll \omega_l$, and then the following approximate expressions for $\omega_\pm$ are useful:

$$\omega_+^2 \approx \omega_l^2 + \omega_c^2 + \frac{\omega_t^2\omega_c^2}{\omega_l^2 + \omega_c^2} \tag{2.22}$$

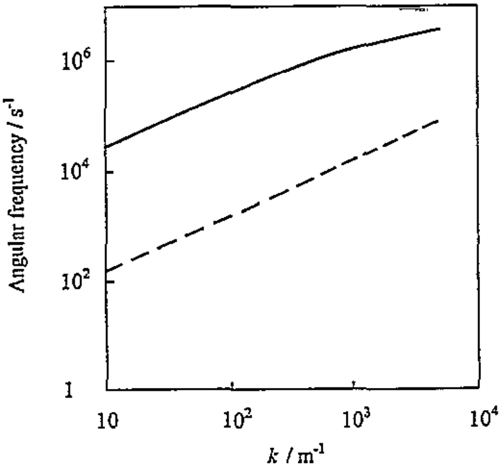


**Figure 1.** The dispersion relations for the crystalline phase *in zero magnetic field: angular frequency*, $\omega$, plotted against wavenumber $k$. $n_0 = 7.46 \times 10^{11}$ m$^{-2}$. Solid line: longitudinal modes; broken line: transverse modes.

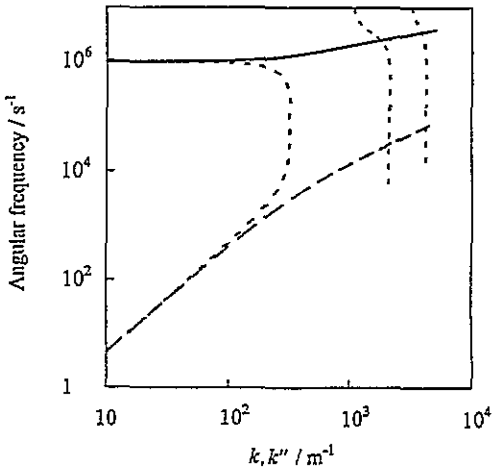


**Figure 2.** The dispersion relations for the crystalline phase *in a vertical magnetic field of* 1.2 T. Angular frequency plotted against $k$ or $k''$: $k$ is a real wavenumber; $k''$ is the magnitude of a purely imaginary wavenumber. Solid line: undamped magnetoplasma wave frequencies plotted against $k$. Broken line: *frequencies of undamped transverse waves plotted* against $k$. Dotted lines: evanescent modes plotted against $k''$.

and

$$\omega_-^2 \approx \frac{\omega_t^2 \omega_l^2}{\omega_l^2 + \omega_c^2}. \tag{2.23}$$

However, to obtain the proper dispersion curves for imaginary $k$ the accurate equation (2.20) must be used. The peak amplitudes in the response, corresponding to the frequencies (2.22) and (2.23), are given respectively by

$$\frac{M}{iek^2\tau}\frac{L_{k,m}}{\Phi_{k,m}} = \frac{\omega_l^2 + \omega_c^2}{\omega_+(\omega_l^2 + 2\omega_c^2)} \quad \text{and} \quad \frac{\omega_c^2 \omega_t}{\omega_+ \omega_l^3}; \tag{2.24}$$

$$\frac{M}{ek^2\tau}\frac{T_{k,m}}{\Phi_{k,m}} = -\frac{\omega_c}{\omega_l^2 + 2\omega_c^2} \quad \text{and} \quad \frac{\omega_c}{\omega_l^2}. \tag{2.25}$$

The linewidths (FWHH) for $L_{k,m}$ and $T_{k,m}$ are equal and given respectively by

$$\frac{1}{\tau}\frac{\omega_l^2 + 2\omega_c^2}{\omega_l^2 + \omega_c^2} \quad \text{and} \quad \frac{1}{\tau}\frac{\omega_l^2}{\omega_l^2 + \omega_c^2}. \tag{2.26}$$

The modes with frequency $\omega_-$ are pure shear modes in the absence of a magnetic field, the effect of the magnetic field being to introduce some longitudinal component and to reduce the frequency (for a given $k$). The modes with frequency $\omega_+$ are longitudinal plasma modes in the absence of the magnetic field, the effect of the magnetic field being to introduce some transverse component and to shift the frequency by an amount that is only weakly dependent on the shear modulus.

Since the system is linear, the response of the pool to a sum of perturbing potentials of the form

$$\Phi = \sum_{k,m} \Phi_{k,m} J_m(kr) \exp(im\theta) \exp(-i\omega t) \tag{2.27}$$

will be a corresponding sum based on the forms (2.12).

If the fields $L$ and $T$ are known then the field $u = (u_r, u_\theta, 0)$ is determined uniquely. It is easily verified that

$$u_r = -\sum_{k,m} \frac{1}{2k} \exp(im\theta)\left[P_{k,m} J_{m-1}(kr) - Q_{k,m} J_{m+1}(kr)\right] \exp(-i\omega t) \tag{2.28}$$

$$u_\theta = -i \sum_{k,m} \frac{1}{2k} \exp(im\theta)\left[P_{k,m} J_{m-1}(kr) + Q_{k,m} J_{m+1}(kr)\right] \exp(-i\omega t) \tag{2.29}$$

where

$$P_{k,m} = L_{k,m} - iT_{k,m} = i\left(\frac{\omega^2 - \omega_t^2 + i\omega/\tau - \omega\omega_c}{\omega\omega_c}\right) T_{k,m} \tag{2.30}$$

and

$$Q_{k,m} = L_{k,m} + iT_{k,m} = i\left(\frac{\omega^2 - \omega_t^2 + i\omega/\tau + \omega\omega_c}{\omega\omega_c}\right) T_{k,m}. \tag{2.31}$$

We have made use of equation (2.19)

In a truly unbounded pool, modes with an imaginary $k$ cannot of course exist. However, such modes can play an important role in a bounded pool, to which we now turn out attention.

## 3. The response of a bounded pool

### *3.1. Boundary conditions*

In general terms, of course, resonant modes can exist in a bounded pool only for a discrete set of frequencies, $\omega$, which are determined by the boundary conditions at the edge of the pool. The calculation of these frequencies is generally rather difficult, because the equilibrium density of the pool is not spatially constant near its edge. Furthermore, the relationship between a perturbation in the density of the pool and the resulting perturbation in the electrostatic potential is generally non-local, with a range of order $h$, and this fact adds to the difficulty near the edge of the pool. The non-local behaviour can be seen from equation (2.8), which shows that the restoring force $-(e/M)\nabla_\perp \phi$ in the equation of motion (2.1) is proportional to the local gradient in the ion density through a factor that is generally dependent on $k$, unless $kh \ll 1$. However, in practice we are often concerned with modes having small wavenumbers, small compared with $1/h$, and we shall argue that we can deal with such modes quite simply by ignoring the *detailed* behaviour near the pool edge and using *effective* boundary conditions. Otherwise it is necessary to use computational techniques, such as those described by Nazin and Shikin (1988) and by Appleyard *et al* (1995).

We shall consider a circular ion pool, of radius $R$, again formed midway between electrodes that are separated by a distance $2h$, and we shall suppose that $h \ll R$, as is the case in practical experiments. As shown in, for example, Glattli *et al* (1985) and Barenghi *et al* (1991), such a pool has an almost constant areal density, $n_0$, except within a distance of order $h$ from its edge, where the density falls gradually to zero, although the pool still has a well defined edge at which the radial component of the density gradient becomes infinite. We shall be interested in modes for which the wavenumber $k$ satisfies the inequality $kh \ll 1$. We shall make the reasonable assumption that such modes propagate in the same way as in an unbounded pool of density $n_0$, and that we can take account of the edge of the pool by simply imposing suitable boundary conditions at $r = R$, the edge of the pool being assumed in effect to be abrupt. This was in fact the approach adopted in Barenghi *et al* (1991), where it was shown from numerical simulations of the behaviour of the pool that one such boundary condition is, to a good approximation,

$$u_r = 0 \qquad \text{at } r = R. \tag{3.1}$$

Taking into account a finite shear modulus, as we do in this paper, we guess that a second boundary condition is

$$\left[r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)\right]_{r=R} = 0 \qquad \text{at } r = R \tag{3.2}$$

which follows from the fact that there can be no shear stress at the edge of the pool.

We emphasize that this approach can describe satisfactorily only those modes that behave smoothly in the region of width $h$ near the edge of the pool; even then it can give only a good approximate description (Glattli *et al* 1985). It cannot describe a mode that involves, for example, any oscillation in the perturbed electrostatic potential in the edge region. It might be thought that such an oscillation could not arise in any mode with a frequency so low as to be comparable with those of the low-$k$ shear modes with which we are particularly concerned in this paper. Curiously this is not true. In particular, it is not true of the types of edge mode considered recently by Nazin and Shikin (1988) and observed by Elliott *et al* (1995b). We shall need to remember the possible existence of these novel edge modes in our later discussion.

### *3.2. Magnetoplasma modes: an elementary treatment*

Before we proceed further we shall summarize the results of existing elementary calculations on the effect of a magnetic field $B$ on the plasma modes in a bounded pool (magnetoplasma modes). The effect of a shear modulus was ignored (even though the calculations were applied to crystalline phases), and use was made of only the boundary condition $u_r = 0$ at an assumed abrupt edge of the pool. As can easily be shown from equations (2.19) and (2.28), this boundary condition leads to allowed values of $k$ that are given by

$$(\omega - \omega_c)mJ_m(kR) - \omega kRJ_{m+1}(kR) = 0. \tag{3.3}$$

We see that axisymmetric and non-axisymmetric modes behave differently. For axisymmetric modes ($m = 0$), which are non-degenerate, the allowed values of $k$ are independent of $B$. As we see from equation (2.22), with $\omega_t = 0$, the frequency of each plasma mode then increases with $B$ according to the relation (Mellor *et al* 1988)

$$\omega^2(B) = \omega^2(0) + \omega_c^2. \tag{3.4}$$

For non-axisymmetric modes, however, the allowed values of $k$ depend on $B$, and this leads to a more complicated behaviour. A degeneracy with respect to the sign of $m$ is removed by the field, and the splitting increases with increasing field. In the case of the lowest modes with a given value of $|m|$, the mode with negative $m$ increases with increasing $B$, while that with positive $m$ decreases. At a sufficiently large value of $B$ the frequency of this latter mode falls below the cyclotron frequency, $\omega_c$. As we see from equation (2.22), $\omega_l$ and therefore $k$ must then become imaginary ($\mathrm{i}k''$). In its radial dependence the mode becomes evanescent, the amplitude being large only near the edge of the pool. The mode is then called an edge magnetoplasma mode (Glattli *et al* 1985, Mast *et al* 1985, Appleyard *et al* 1995). With further increases in $B$ the value of $k''$ will also increase, leading eventually to a failure of the assumption that $k''h \ll 1$. Our treatment must therefore fail in the high-field limit. There is then the need for a treatment such as that used by Nazin and Shikin (1988). As we have already mentioned, this treatment leads to the prediction that there are families of (non-axisymmetric) edge modes other than those considered here and for which the perturbed electrostatic potential oscillates in space within the region of width $h$ close to the edge of the pool. Surprisingly, as we have already mentioned, such modes can have low frequencies.

### 3.3. *Use of the correct boundary conditions in the crystal phase*

We emphasize that the calculation mentioned in subsection 3.2 ignored any shear modulus in the system and ignored the associated boundary condition (3.2). We now examine the effect of taking these features of the real crystalline system into account.

Consider first the case when there is no magnetic field. Equations (2.13) and (2.14) are uncoupled, and we have pure plasma modes ($T = 0$) or pure shear modes ($L = 0$). For axisymmetric modes ($m = 0$) the problem is then straightforward. For the plasma modes, $u_\theta = 0$ everywhere, while for the shear modes $u_r = 0$ everywhere, as we see from equations (2.28) and (2.29). The wavenumbers and frequencies of the plasma modes are determined by the boundary condition (3.1); those of the shear modes by (3.2). However, this simple situation does not hold for non-axisymmetric modes. In this case both $u_r$ and $u_\theta$ are generally finite, and each mode must satisfy *both* (3.1) and (3.2). A single mode, with a single value of $k$, cannot do this; if the mode satisfies (3.1) it generally violates (3.2), and *vice versa*. The correct modes must therefore be mixtures of two 'free space' modes, with different $k$ but the same frequency $\omega$. In the case of non-axisymmetric modes in zero magnetic field, one of these modes will be a plasma mode and the other a shear mode, as we see from figure 1.

### 3.4. *Normal modes in a magnetic field*

In the presence of a magnetic field the situation is more complicated. Consider first the case of axisymmetric modes. If we examine figure 2 we see that a mode with, for example, a low frequency (less than $\omega_c$) and real wavenumber can mix only with a mode with an imaginary wavenumber, $k = \mathrm{i}k''$. Owing to the multivalued nature of $\tanh(\mathrm{i}k''h) = \mathrm{i}\tan(k''h)$, there appear to be many modes with imaginary wavenumbers that could be involved in the mixing. However, all except that with the smallest imaginary wavenumber certainly violate the condition $k''h \ll 1$. Therefore our treatment of them cannot be correct; indeed they may not exist. Tentatively we shall ignore them. In the case of non-axisymmetric modes the situation becomes even more complicated, owing to the existence of the Nazin–Shikin modes, which might also play a role. Again, however, we shall tentatively ignore them. Our approach then is to assume that we can make use of effective boundary conditions, represented by equations (3.1) and (3.2), and to assume also that we need mix only modes for which the real and imaginary parts of the wavenumber are small compared with $1/h$.

With these understandings we can define the normal modes of the pool. A normal mode of frequency $\omega$ will be of the form

$$L(r,\theta)\exp(-\mathrm{i}\omega t)=\left[L_{k_1,m}J_m(k_1r)+L_{k_2,m}J_m(k_2r)\right]\exp(\mathrm{i}m\theta)\exp(-\mathrm{i}\omega t) \tag{3.5}$$

$$T(r,\theta)\exp(-\mathrm{i}\omega t)=\left[T_{k_1,m}J_m(k_1r)+T_{k_2,m}J_m(k_2r)\right]\exp(\mathrm{i}m\theta)\exp(-\mathrm{i}\omega t) \tag{3.6}$$

where $k_1(\omega)$ and $k_2(\omega)$ are solutions of the equation

$$\left[\omega^2-\omega_t^2(k)\right]\left[\omega^2-\omega_l^2(k)\right]-\omega^2\omega_c^2=0. \tag{3.7}$$

The coefficients $T_{k_1,m}$ and $T_{k_2,m}$ are related to $L_{k_1,m}$ and $L_{k_2,m}$ respectively by equation (2.19). The wavenumbers $k_1(\omega)$ and $k_2(\omega)$ correspond to the intersection of a horizontal line at frequency $\omega$ with the dispersion curves of figure 2 (we neglect intersections at high wavenumbers, as we have already explained). For $\omega < \omega_c$, one of the wavenumbers $k_1$ and $k_2$ is real and the other imaginary; for $\omega > \omega_c$, both are real. These wavenumbers must be chosen in order that (3.5) and (3.6) lead to displacements $u_r$ and $u_\theta$ satisfying the boundary conditions (3.1) and (3.2). This procedure defines sets of wavenumbers $k_{1,i}$ and $k_{2,i}$ ($i = 1, 2, 3\ldots$), corresponding frequencies $\omega_i$, and ratios $L_{k_{1,i},m}/L_{k_{2,i},m}$, which, when substituted into (3.5) and (3.6), define the required normal modes, $[L_{i,m}(r,\theta), T_{i,m}(r,\theta)]$. The frequencies of the low-lying shear modes for a typical pool of positive ions ($M = 2.02\times10^{-25}$ kg) are given in table 1, where they are compared with those obtained by ignoring the boundary condition (3.1). In deriving these frequencies, and in calculating the response shown later in figure 3, we have assumed that the shear modulus is given by equation (1.2), with a small correction for the finite temperature obtained from the experimental data of Glattli *et al* (1985) on a two-dimensional electron solid.

**Table 1.** The frequencies (in Hz) of the low-lying shear modes of a circular pool of radius 13.6 mm; $n_0 = 7.46\times10^{11}$ m$^{-2}$. Positive ions; $B = 1.2$ T; $T = 17$ mK. For a given azimuthal quantum number $m$ the modes are labelled by successive integers $n$. The figures in brackets are those obtained if the boundary condition (3.1) is ignored.

| Mode $m$ | 0 | 1 | −1 | 2 | −2 |
|---|---|---|---|---|---|
| $n$ | | | | | |
| 1 | 709.4 | 348.1 | 348.2 | 606.3 | 606.3 |
| | (673.9) | (318.0) | (317.1) | (577.0) | (575.9) |
| 2 | 1305.8 | 978.6 | 979.3 | 1248.4 | 1248.7 |
| | (1289.7) | (970.4) | (970.2) | (1241.4) | (1241.1) |
| 3 | 1890.0 | 1581.3 | 1581.9 | 1851.2 | 1851.5 |
| | (1881.4) | (1578.3) | (1578.2) | (1848.5) | (1848.3) |

In the absence of a magnetic field, modes with equal values of $|m|$ are degenerate. In the case of the longitudinal plasma modes a magnetic field of order 1 T leads to a large splitting (Glattli *et al* 1985, Appleyard *et al* 1995). We see from table 1, however, that for the shear modes the splitting is very small (less than 0.1%), and it is much too small to have been resolved in experiments so far carried out (Elliott *et al* 1995a).

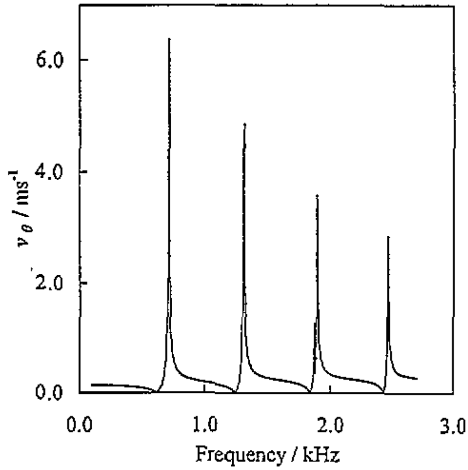


**Figure 3.** The calculated response to an axisymmetric edge drive of a pool of radius 13.6 mm at a temperature of 17 mK. $n_0 = 7.46 \times 10^{11}$ m$^{-2}$; $B = 1.2$ T. The edge is driven at an amplitude of 0.29 $\mu$m. The response $v_\theta$ is the $\theta$-component of the velocity measured at the edge of the pool. $\tau = 19.2$ ms (see text). The peaks correspond to the first four axisymmetric transverse modes.

### *3.5. The response of the pool to an external drive*

We are interested in the response of the pool to being driven at a frequency $\omega$ close to the natural frequency of one of the modes, in the presence of a small but finite damping, described by the relaxation time $\tau$. In principle, two forms of drive can be used: the bulk of the pool can be exposed to an electric field oscillating at frequency $\omega$ ('bulk drive'); or the edge of the pool can be moved to and fro at the frequency $\omega$ ('edge drive'). In practice both types of drive are present at the same time.

We consider first the response of the pool to a *bulk drive* described by an electrostatic potential in the plane of the pool equal to

$$\Phi(r, \theta) \exp(-\mathrm{i}\omega t). \tag{3.8}$$

The normal modes $L_{i,m}(r, \theta)$ form a complete set, and therefore we can write

$$\Phi(r, \theta) = \sum_{i,m} \gamma_{i,m} L_{i,m}(r, \theta) \tag{3.9}$$

where the coefficients $\gamma_{i,m}$ are determined in the usual way. Suppose that $\omega$ is close to the frequency, $\omega_j$, of one of the normal modes $[L_{j,m}(r, \theta), T_{j,m}(r, \theta)]$, and that the damping is small. Then, to a good approximation, only this mode will be excited with a significant amplitude, and only the component $\gamma_{j,m} L_{j,m}(r, \theta)$ in the summation (3.9) need be taken into account. Using (3.5) we see that the driving potential can be written as a linear combination of two terms

$$\gamma_{j,m}\left[L_{k_{1,j},m} J_m(k_{1,j} r) + L_{k_{2,j},m} J_m(k_{2,j} r)\right] \exp(\mathrm{i}m\theta). \tag{3.10}$$

The response can then be calculated from equations (2.18) and (2.19); it will be a linear combination of two terms, due to the two terms in the driving potential (3.10).

The case of an *edge drive* involves the replacement of the boundary condition (3.1) by one that describes the imposed motion of the edge of the pool. We shall consider the case where this motion is described by the boundary condition

$$u_r = U_0 \exp(\mathrm{i}m\theta)\exp(-\mathrm{i}\omega t) \qquad \text{at } r = R \tag{3.11}$$

so that only modes with a particular value of $m$ are excited (the more general case can be treated by straightforward superposition). We must recognize that with an edge drive waves are generated at the edge of the pool and propagate inwards through the pool centre and out again. We must therefore take account of the spatial attenuation of these waves as they propagate, which means that the wavenumbers we use must be complex, rather than simply pure real or pure imaginary. These complex wavenumbers will be given by the vanishing of the denominator of equation (2.18). We are particularly interested in the case when $\omega \ll \omega_c$; the two wavenumbers are then of the form

$$k_a = k_0 + \mathrm{i}k'' \qquad k_b = \mathrm{i}\tilde{k}_0 + k' \tag{3.12}$$

where $k'' \ll k_0$ and $k' \ll \tilde{k}_0$. Then the response of the pool has the form

$$u_r = -\frac{1}{2}\sum_{k_a,k_b}\left[\frac{1}{k_a}\left\{P_{k_a,m}J_{m-1}(k_a r) - Q_{k_a,m}J_{m+1}(k_a r)\right\} + \frac{1}{k_b}\left\{P_{k_b,m}J_{m-1}(k_b r) - Q_{k_b,m}J_{m+1}(k_b r)\right\}\right]\exp(\mathrm{i}m\theta)\exp(-\mathrm{i}\omega t) \tag{3.13}$$

$$u_\theta = -\frac{\mathrm{i}}{2}\sum_{k_a,k_b}\left[\frac{1}{k_a}\left\{P_{k_a,m}J_{m-1}(k_a r) + Q_{k_a,m}J_{m+1}(k_a r)\right\} + \frac{1}{k_b}\left\{P_{k_b,m}J_{m-1}(k_b r) + Q_{k_b,m}J_{m+1}(k_b r)\right\}\right]\exp(\mathrm{i}m\theta)\exp(-\mathrm{i}\omega t) \tag{3.14}$$

where the Ps and Qs are given in terms of parameters $T_{k_a,m}$ and $T_{k_b,m}$ by equations (2.30) and (2.31). Applying the boundary conditions (3.1) and (3.2) allows us to determine $T_{k_a,m}$ and $T_{k_b,m}$, and so obtain the required response.

In figure 3 we show an example of the calculated response of the pool described in table 1 to an axisymmetric edge drive over a range of low frequencies. The pool is in the crystal phase at 17 mK, and the modes shown are the lowest four axisymmetric shear modes in a field of 1.2 T. It is assumed that the modes are damped only by the drag on a moving ion associated with the observed ripplon-limited mobility at the temperature of 17 mK and a trapping depth of 55.3 nm. In reality other contributions to the damping may exist; for example, from internal friction in the two-dimensional crystal. The results of this type of calculation are in good agreement with experiment (Elliott *et al* 1995a).

### *3.6. The response in the fluid phase*

As we explained in the introduction, we can obtain the response in the fluid phase by replacing the shear modulus $\mu$ by $-\mathrm{i}\omega\eta$, where $\eta$ is the fluid viscosity. At low frequencies the modes of the pool that are excited are now viscous waves, modified by the damping associated with the relaxation time $\tau$, so that there is no longer any resonant response. We show in figure 4 examples of the calculated response to an axisymmetric edge drive of the ion pool of table 1 at a temperature of 200 mK, which is about 4 mK above its

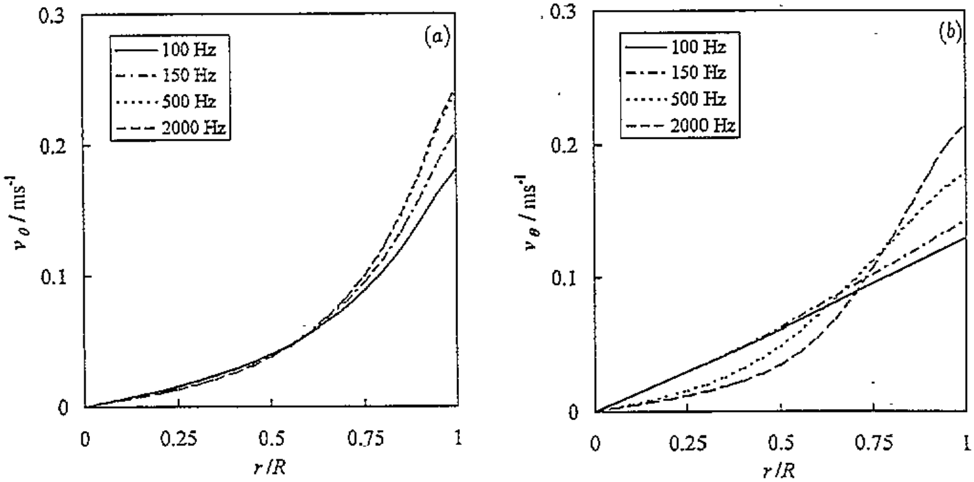


**Figure 4.** The calculated responses to an axisymmetric edge drive of the same pool as in figure 3, but at a temperature of 200 mK. $B = 1.2$ T. The edge is again driven at an amplitude of 0.29 $\mu$m. $\tau = 1.72$ ms (see text). The graphs show the $\theta$-component of the velocity plotted against radial distance from the centre of the pool for different frequencies. (a) kinematic viscosity, $\nu = \eta/Mn_0 = 2.20 \times 10^{-7}$ m$^2$ s$^{-1}$; (b) $\nu = 2.20 \times 10^{-2}$ m$^2$ s$^{-1}$. It is assumed that the system is behaving as a conventional two-dimensional fluid.

melting temperature. The trapping depth is again 55.3 nm. We assume that the pool is in a conventional fluid phase, and we show plots of the amplitude of the $\theta$-component of the ionic velocity against radial distance from the centre of the pool for various frequencies and two values of the kinematic viscosity of the ionic fluid. The value of $\tau$ is chosen to correspond to the observed ripplon-limited mobility at the temperature and trapping depth concerned. We shall compare these calculations with experiment in a later publication.

## 4. Summary and conclusions

We have reported a theoretical study of the response to an oscillating electric potential of a two-dimensional pool of ions trapped below the free surface of superfluid helium in the presence of a vertical magnetic field, the driving field and the response being in the plane of the ions. We have concentrated on modes of response of long wavelength, and our results underly the experimental study of the modes of transverse oscillation of the ion pools in both the crystal and the fluid phases. The results are applicable to other two-dimensional classical Coulomb systems.

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